

# ON SEMITOPOLOGICAL BICYCLIC EXTENSIONS OF LINEARLY ORDERED GROUPS

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**ABSTRACT.** For a linear ordered group  $G$  the natural partial order and solutions of equations on the semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  are described. We study topologizations of the semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$ . In particular we show that for an arbitrary countable linearly ordered group  $G$  every Baire  $T_1$ -topology  $\tau$  on  $\mathcal{B}(G)$  ( $\mathcal{B}^+(G)$ ) such that  $(\mathcal{B}(G), \tau)$  ( $(\mathcal{B}^+(G), \tau)$ ) is a semitopological semigroup is discrete. Also we prove that for an arbitrary linearly non-densely ordered group  $G$  every Hausdorff topology  $\tau$  on the semigroup  $\mathcal{B}(G)$  ( $\mathcal{B}^+(G)$ ) such that  $(\mathcal{B}(G), \tau)$  ( $(\mathcal{B}^+(G), \tau)$ ) is a semitopological semigroup is discrete.

## 1. INTRODUCTION AND PRELIMINARIES

We shall follow the terminology of [13, 16, 18, 29, 36, 37].

A *semigroup* is a non-empty set with a binary associative operation. A semigroup  $S$  is called *inverse* if for any  $x \in S$  there exists a unique  $y \in S$  such that  $x \cdot y \cdot x = x$  and  $y \cdot x \cdot y = y$ . Such an element  $y$  in  $S$  is called the *inverse* of  $x$  and denoted by  $x^{-1}$ . The map defined on an inverse semigroup  $S$  which maps every element  $x$  of  $S$  to its inverse  $x^{-1}$  is called the *inversion*.

If  $S$  is a semigroup, then we shall denote the subset of idempotents in  $S$  by  $E(S)$ . If  $S$  is an inverse semigroup, then  $E(S)$  is closed under multiplication and we shall refer to  $E(S)$  as the *band of  $S$* . If the band  $E(S)$  is a non-empty subset of  $S$ , then the semigroup operation on  $S$  determines the following partial order  $\preceq$  on  $E(S)$ :  $e \preceq f$  if and only if  $ef = fe = e$ . This order is called the *natural partial order* on  $E(S)$ . A *semilattice* is a commutative semigroup of idempotents.

Let  $\mathcal{I}_X$  denote the set of all partial one-to-one transformations of an infinite set  $X$  together with the following semigroup operation:  $x(\alpha\beta) = (x\alpha)\beta$  if  $x \in \text{dom}(\alpha\beta) = \{y \in \text{dom } \alpha : y\alpha \in \text{dom } \beta\}$ , for  $\alpha, \beta \in \mathcal{I}_X$ . The semigroup  $\mathcal{I}_X$  is called the *symmetric inverse semigroup* over the set  $X$  (see [16]). The symmetric inverse semigroup was introduced by Wagner [38] and it plays a major role in the theory of semigroups.

The bicyclic semigroup  $\mathcal{C}(p, q)$  is the semigroup with the identity 1 generated by two elements  $p$  and  $q$  subjected only to the condition  $pq = 1$ . The bicyclic monoid is a combinatorial bisimple  $F$ -inverse semigroup and it plays an important role in the algebraic theory of semigroups and in the theory of topological semigroups. For example the well-known O. Andersen's result [1] states that a (0-) simple semigroup is completely (0-) simple if and only if it does not contain the bicyclic semigroup. The bicyclic semigroup does not embed into stable semigroups [31].

Recall from [22] that a *partially-ordered group* is a group  $(G, \cdot)$  equipped with a partial order  $\leq$  that is translation-invariant; in other words, the binary relation  $\leq$  has the property that, for all  $a, b, g \in G$ , if  $a \leq b$  then  $a \cdot g \leq b \cdot g$  and  $g \cdot a \leq g \cdot b$ .

Later by  $e$  we denote the identity of a group  $G$ . The set  $G^+ = \{x \in G : e \leq x\}$  in a partially ordered group  $G$  is called the *positive cone*, or the *integral part*, of  $G$  and satisfies the properties:

- 1)  $G^+ \cdot G^+ \subseteq G^+$ ;    2)  $G^+ \cap (G^+)^{-1} = \{e\}$ ;    and    3)  $x^{-1} \cdot G^+ \cdot x \subseteq G^+$  for each  $x \in G$ .

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Any subset  $P$  of a group  $G$  that satisfies the conditions 1)–3) induces a partial order on  $G$  ( $x \leq y$  if and only if  $x^{-1} \cdot y \in P$ ) for which  $P$  is the positive cone. An elements of the set  $G^+ \setminus \{e\}$  is called *positive*.

A *linearly ordered* or *totally ordered group* is an ordered group  $G$  such that the order relation “ $\leq$ ” is total [12].

In the remainder we shall assume that  $G$  is a linearly ordered group.

For every  $g \in G$  we denote

$$G^+(g) = \{x \in G : g \leq x\}.$$

The set  $G^+(g)$  is called a *positive cone on element  $g$*  in  $G$ .

For arbitrary elements  $g, h \in G$  we consider a partial map  $\alpha_h^g : G \rightarrow G$  defined by the formula

$$(x)\alpha_h^g = x \cdot g^{-1} \cdot h, \quad \text{for } x \in G^+(g).$$

We observe that Lemma XIII.1 from [12] implies that for such partial map  $\alpha_h^g : G \rightarrow G$  the restriction  $\alpha_h^g : G^+(g) \rightarrow G^+(h)$  is a bijective map.

We denote

$$\mathcal{B}(G) = \{\alpha_h^g : G \rightarrow G : g, h \in G\} \text{ and } \mathcal{B}^+(G) = \{\alpha_h^g : G \rightarrow G : g, h \in G^+\},$$

and consider on the sets  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  the operation of the composition of partial maps. Simple verifications show that

$$(1) \quad \alpha_h^g \cdot \alpha_l^k = \alpha_b^a, \quad \text{where } a = (h \vee k) \cdot h^{-1} \cdot g \quad \text{and} \quad b = (h \vee k) \cdot k^{-1} \cdot l,$$

for  $g, h, k, l \in G$ . Therefore, property 1) of the positive cone and condition (1) imply that  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  are subsemigroups of  $\mathcal{I}_G$ .

By Proposition 1.2 from [24] for a linearly ordered group  $G$  the following assertions hold:

- (i) elements  $\alpha_h^g$  and  $\alpha_g^h$  are inverses of each other in  $\mathcal{B}(G)$  for all  $g, h \in G$  (resp.,  $\mathcal{B}^+(G)$  for all  $g, h \in G^+$ );
- (ii) an element  $\alpha_h^g$  of the semigroup  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ) is an idempotent if and only if  $g = h$ ;
- (iii)  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  are inverse subsemigroups of  $\mathcal{I}_G$ ;
- (iv) the semigroup  $\mathcal{B}(G)$  (resp.,  $\mathcal{B}^+(G)$ ) is isomorphic to the set  $S_G = G \times G$  (resp.,  $S_G^+ = G^+ \times G^+$ ) with the following semigroup operation:

$$(2) \quad (a, b)(c, d) = \begin{cases} (c \cdot b^{-1} \cdot a, d), & \text{if } b < c; \\ (a, d), & \text{if } b = c; \\ (a, b \cdot c^{-1} \cdot d), & \text{if } b > c, \end{cases}$$

where  $a, b, c, d \in G$  (resp.,  $a, b, c, d \in G^+$ ).

It is obvious that:

- (1) if  $G$  is isomorphic to the additive group of integers  $(\mathbb{Z}, +)$  with usual linear order  $\leq$  then the semigroup  $\mathcal{B}^+(G)$  is isomorphic to the bicyclic semigroup  $\mathcal{C}(p, q)$  and the semigroup  $\mathcal{B}(G)$  is isomorphic to the extended bicyclic semigroup  $\mathcal{C}_{\mathbb{Z}}$  (see [19]);
- (2) if  $G$  is the additive group of real numbers  $(\mathbb{R}, +)$  with usual linear order  $\leq$  then the semigroup  $\mathcal{B}(G)$  is isomorphic to  $B_{(-\infty, \infty)}^2$  (see [32, 33]) and the semigroup  $\mathcal{B}^+(G)$  is isomorphic to  $B_{[0, \infty)}^1$  (see [2, 3, 4, 5, 6]), and
- (3) the semigroup  $\mathcal{B}^+(G)$  is isomorphic to the semigroup  $S(G)$  which is defined in [20, 21].

In the paper [24] semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  are studied for a linearly ordered group  $G$ . There described Green's relations on  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$ , their bands and showed that they are bisimple. Also in [24] proved that for a commutative linearly ordered group  $G$  all non-trivial congruences on the semigroup  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  are group congruences if and only if the group  $G$  is archimedean and described the structure of group congruences on the semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$ .

Later in this paper we identify the semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  with the semigroups  $S_G$  and  $S_G^+$ , respectively, with multiplications defined by formula (2).

We recall that a topological space  $X$  is said to be

- *locally compact*, if every point  $x \in X$  has an open neighbourhood with the compact closure;
- *Čech-complete*, if  $X$  Tychonoff and the Čech-Stone compactification  $\beta X$  of  $X$  has the  $F_\sigma$  remainder;
- *Baire*, if for each sequence  $A_1, A_2, \dots, A_i, \dots$  of nowhere dense subsets of  $X$  the union  $\bigcup_{i=1}^\infty A_i$  is a co-dense subset of  $X$ .

Every Hausdorff locally compact space is Čech-complete, and every Čech-complete space is Baire (see [18]).

A *(semi)topological semigroup* is a topological space with a (separately) continuous semigroup operation. A topological semigroup which is an inverse semigroup is called an *inverse topological semigroup*. A *topological inverse semigroup* is an inverse topological semigroup with continuous inversion. A topology  $\tau$  on a (inverse) semigroup  $S$  is called *(inverse) semigroup* if  $(S, \tau)$  is a topological (inverse) semigroup.

The bicyclic semigroup admits only the discrete semigroup topology and if a topological semigroup  $S$  contains it as a dense subsemigroup then  $\mathcal{C}(p, q)$  is an open subset of  $S$  [17]. Bertman and West in [11] extend this result for the case of Hausdorff semitopological semigroups. Stable and  $\Gamma$ -compact topological semigroups do not contain the bicyclic semigroup [7, 30]. The problem of an embedding of the bicyclic monoid into compact-like topological semigroups studied in [8, 9, 26]. Also in the paper [19] proved that the discrete topology is the unique topology on the extended bicyclic semigroup  $\mathcal{C}_\mathbb{Z}$  such that the semigroup operation on  $\mathcal{C}_\mathbb{Z}$  is separately continuous. Also, for many (0-) bisimple semigroups of transformations  $S$  the following statement holds: *every Hausdorff Baire (in particular locally compact) topology  $\tau_c$  on  $S$  such that  $(S, \tau_c)$  is a semitopological group is discrete* (see [14, 15, 25, 27, 28]). In the paper [35] Mesyan, Mitchell, Morayne and Péresse showed that if  $E$  is a finite graph, then the only locally compact Hausdorff semigroup topology on the graph inverse semigroup  $G(E)$  is the discrete topology. In [10] was proved that the conclusion of this statement also holds for graphs  $E$  consisting of one vertex and infinitely many loops (i.e., infinitely-generated polycyclic monoids). Amazing dichotomy for the bicyclic monoid with adjoined zero  $\mathcal{C}^0 = \mathcal{C}(p, q) \sqcup \{0\}$  was proved in [23]: every Hausdorff locally compact semitopological bicyclic semigroup with adjoined zero  $\mathcal{C}^0$  is either compact or discrete.

For a linear ordered group  $G$  the natural partial order and solutions of equations on the semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  are described. We study topologizations of the semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$ . In particular we show that for an arbitrary countable linearly ordered group  $G$  every Baire  $T_1$ -topology  $\tau$  on  $\mathcal{B}(G)$  ( $\mathcal{B}^+(G)$ ) such that  $(\mathcal{B}(G), \tau)$  ( $(\mathcal{B}^+(G), \tau)$ ) is a semitopological semigroup is discrete. Also we prove that for an arbitrary linearly non-densely every Hausdorff topology  $\tau$  on the semigroup  $\mathcal{B}(G)$  ( $\mathcal{B}^+(G)$ ) such that  $(\mathcal{B}(G), \tau)$  ( $(\mathcal{B}^+(G), \tau)$ ) is a semitopological semigroup is discrete, and hence  $(\mathcal{B}(G), \tau)$  ( $(\mathcal{B}^+(G), \tau)$ ) is a discrete subspace of any Hausdorff semitopological semigroup which contains  $\mathcal{B}(G)$  ( $\mathcal{B}^+(G)$ ) as a subsemigroup.

## 2. SOLUTIONS OF SOME EQUATIONS AND THE NATURAL PARTIAL ORDER ON THE SEMIGROUPS $\mathcal{B}(G)$ AND $\mathcal{B}^+(G)$

It is well known that every inverse semigroup  $S$  admits the *natural partial order*:

$$s \preceq t \quad \text{if and only if} \quad s = et \quad \text{for some} \quad e \in E(S).$$

This order induces the natural partial order on the semilattice  $E(S)$ , and for arbitrary  $s, t \in S$  the following conditions are equivalent:

$$(3) \quad (\alpha) \ s \preceq t; \quad (\beta) \ s = ss^{-1}t; \quad (\gamma) \ s = ts^{-1}s,$$

(see [34, Chapter 3].)

Later we need the following lemma, which describes the natural partial order on the semigroup  $\mathcal{B}(G)$ :

**Lemma 2.1.** *Let  $G$  be a linearly ordered group. Then for arbitrary elements  $(a, b), (c, d) \in \mathcal{B}(G)$  the following conditions are equivalent:*

- (i)  $(a, b) \preceq (c, d)$  in  $\mathcal{B}(G)$ ;
- (ii)  $a^{-1} \cdot b = c^{-1} \cdot d$  and  $a \geq c$  in  $G$ ;

(iii)  $b^{-1} \cdot a = d^{-1} \cdot b$  and  $b \geq d$  in  $G$ .

*Proof.* (i)  $\Rightarrow$  (ii) The equivalence of conditions  $(\alpha)$  and  $(\beta)$  in (3) implies that  $(a, b) \preceq (c, d)$  in  $\mathcal{B}(G)$  if and only if  $(a, b) = (a, b)(a, b)^{-1}(c, d)$ . Therefore we have that

$$(a, b) = (a, b)(a, b)^{-1}(c, d) = (a, b)(b, a)(c, d) = (a, a)(c, d) = \begin{cases} (c \cdot a^{-1} \cdot a, d), & \text{if } a < c; \\ (c, d), & \text{if } a = c; \\ (a, a \cdot c^{-1} \cdot d), & \text{if } a > c. \end{cases}$$

This implies that

$$(a, b) = \begin{cases} (c, d), & \text{if } a < c; \\ (c, d), & \text{if } a = c; \\ (a, a \cdot c^{-1} \cdot d), & \text{if } a > c, \end{cases}$$

and hence the condition  $(a, b) \preceq (c, d)$  in  $\mathcal{B}(G)$  implies that  $a^{-1} \cdot b = c^{-1} \cdot d$  and  $a \geq c$  in  $G$ .

(ii)  $\Rightarrow$  (i) Fix arbitrary  $(a, b), (c, d) \in \mathcal{B}(G)$  such that  $a^{-1} \cdot b = c^{-1} \cdot d$  and  $a \geq c$  in  $G$ . Then we have that

$$(a, b)(a, b)^{-1}(c, d) = (a, b)(b, a)(c, d) = (a, a)(c, d) = (a, a \cdot c^{-1} \cdot d) = (a, b),$$

and hence  $(a, b) \preceq (c, d)$  in  $\mathcal{B}(G)$ .

The proof of the equivalence (ii)  $\Leftrightarrow$  (iii) is trivial.  $\square$

The definition the semigroup operation in  $\mathcal{B}(G)$  implies that  $(a, b) = (a, c)(c, d)(d, b)$  for arbitrary elements  $a, b, c, d$  of the group  $G$ . The following two propositions give amazing descriptions of solutions of some equations in the semigroup  $\mathcal{B}(G)$ .

**Proposition 2.2.** *Let  $a, b, c, d$  be arbitrary elements of a linearly ordered group  $G$ . Then the following conditions hold:*

- (i)  $(a, b) = (a, c)(x, y)$  for  $(x, y) \in \mathcal{B}(G)$  if and only if  $(c, b) \preceq (x, y)$  in  $\mathcal{B}(G)$ ;
- (ii)  $(a, b) = (x, y)(d, b)$  for  $(x, y) \in \mathcal{B}(G)$  if and only if  $(a, d) \preceq (x, y)$  in  $\mathcal{B}(G)$ ;
- (iii)  $(a, b) = (a, c)(x, y)(d, b)$  for  $(x, y) \in \mathcal{B}(G)$  if and only if  $(c, d) \preceq (x, y)$  in  $\mathcal{B}(G)$ .

*Proof.* (i)  $(\Rightarrow)$  Suppose that  $(a, b) = (a, c)(x, y)$  for some  $(x, y) \in \mathcal{B}(G)$ . Then we have that

$$(a, c)(x, y) = \begin{cases} (a, c \cdot x^{-1} \cdot y), & \text{if } c > x; \\ (a, y), & \text{if } c = x; \\ (x \cdot c^{-1} \cdot a, y), & \text{if } c < x. \end{cases}$$

Then in the case when  $c > x$  we get that  $b = c \cdot x^{-1} \cdot y$  and hence Lemma 2.1 implies that  $(c, b) \preceq (x, y)$  in  $\mathcal{B}(G)$ . Also, in the case when  $c = x$  we have that  $b = y$ , which implies the inequality  $(c, b) \preceq (x, y)$  in  $\mathcal{B}(G)$ . Case  $c < x$  does not hold because the group operation on  $G$  implies that  $x \cdot c^{-1} \cdot a < a$ .

$(\Leftarrow)$  Suppose that the relation  $(c, b) \preceq (x, y)$  holds in  $\mathcal{B}(G)$ . Then by Lemma 2.1 we have that  $c^{-1} \cdot b = x^{-1} \cdot y$  and  $c \geq x$  in  $G$ , and hence the semigroup operation of  $\mathcal{B}(G)$  implies that

$$(a, c)(x, y) = (a, c \cdot x^{-1} \cdot y) = (a, c \cdot c^{-1} \cdot b) = (a, b).$$

The proof of statement (ii) is similar to statement (i).

(iii)  $(\Rightarrow)$  Suppose that  $(a, b) = (a, c)(x, y)(d, b)$  for some  $(x, y) \in \mathcal{B}(G)$ . Then we have that

$$(a, c)(x, y) = \begin{cases} (a, c \cdot x^{-1} \cdot y), & \text{if } c > x; \\ (a, y), & \text{if } c = x; \\ (x \cdot c^{-1} \cdot a, y), & \text{if } c < x. \end{cases}$$

Therefore,

(a) if  $c > x$  then

$$(a, c)(x, y)(d, b) = (a, c \cdot x^{-1} \cdot y)(d, b) = \begin{cases} (a, c \cdot x^{-1} \cdot y \cdot d^{-1} \cdot b), & \text{if } c \cdot x^{-1} \cdot y > d; \\ (a, b), & \text{if } c \cdot x^{-1} \cdot y = d; \\ (d \cdot y^{-1} \cdot x \cdot c^{-1} \cdot a, b), & \text{if } c \cdot x^{-1} \cdot y < d, \end{cases}$$

(b) if  $c = x$  then

$$(a, c)(x, y)(d, b) = (a, y)(d, b) = \begin{cases} (a, y \cdot d^{-1} \cdot b), & \text{if } y > d; \\ (a, b), & \text{if } y = d; \\ (d \cdot y^{-1} \cdot a, b), & \text{if } y < d, \end{cases}$$

(c) if  $c < x$  then

$$(a, c)(x, y)(d, b) = (x \cdot c^{-1} \cdot a, y)(d, b) = \begin{cases} (x \cdot c^{-1} \cdot a, y \cdot d^{-1} \cdot b), & \text{if } y > d; \\ (x \cdot c^{-1} \cdot a, b), & \text{if } y = d; \\ (d \cdot y^{-1} \cdot x \cdot c^{-1} \cdot a, b), & \text{if } y < d. \end{cases}$$

Then the equality  $(a, b) = (a, c)(x, y)(d, b)$  implies that

(a) if  $c > x$  then  $c \cdot x^{-1} \cdot y \cdot d^{-1} = e$  in  $G$ ;

(b) if  $c = x$  then  $y = d$ ;

and case (c) does not hold. Hence by Lemma 2.1 we get that  $(c, d) \preccurlyeq (x, y)$  in  $\mathcal{B}(G)$ .

( $\Leftarrow$ ) Suppose that the relation  $(c, d) \preccurlyeq (x, y)$  holds in  $\mathcal{B}(G)$ . Then by Lemma 2.1 we have that  $c^{-1} \cdot d = x^{-1} \cdot y$  and  $c \geq x$  in  $G$ , and hence the semigroup operation of  $\mathcal{B}(G)$  implies that

$$\begin{aligned} (a, c)(x, y)(d, b) &= (a, c)(x, y)(c \cdot x^{-1} \cdot y, b) = \\ &= (a, c)(c \cdot x^{-1} \cdot y \cdot y^{-1} \cdot x, b) = \\ &= (a, c)(c \cdot x^{-1} \cdot x, b) = \\ &= (a, c)(c, b) = \\ &= (a, b), \end{aligned}$$

because  $c \cdot x^{-1} \cdot y \geq y$  in  $G$ . □

**Proposition 2.3.** *Let  $a, b, c, d$  be elements of a linearly ordered group  $G$ . Then the following statements hold:*

- (i) if  $a < c$  in  $G$  then the equation  $(a, b) = (c, d)(x, y)$  has no solutions in  $\mathcal{B}(G)$ ;
- (ii) if  $a > c$  in  $G$  then the equation  $(a, b) = (c, d)(x, y)$  has the unique solution  $(x, y) = (a \cdot c^{-1} \cdot d, b)$  in  $\mathcal{B}(G)$ ;
- (iii) if  $b < d$  in  $G$  then the equation  $(a, b) = (x, y)(c, d)$  has no solutions in  $\mathcal{B}(G)$ ;
- (iv) if  $b > d$  in  $G$  then the equation  $(a, b) = (x, y)(c, d)$  has the unique solution  $(x, y) = (a, b \cdot d^{-1} \cdot c)$  in  $\mathcal{B}(G)$ .

*Proof.* (i) Assume that  $a < c$ . Then formula (2) implies that  $d < x$  in  $G$  and hence  $(a, b) = (x \cdot d^{-1} \cdot c, y)$ . This implies that  $a = x \cdot d^{-1} \cdot c$  and  $b = y$ . Since  $d < x$ , the equality  $a = x \cdot d^{-1} \cdot c$  implies that  $a > c$ , which contradicts the assumption of statement (i).

(ii) Assume that  $a > c$ . Then formula (2) implies that  $d < x$  in  $G$  and hence we have that  $(a, b) = (x \cdot d^{-1} \cdot c, y)$ . This implies the following equalities  $x = a \cdot c^{-1} \cdot d$  and  $y = b$ .

The proof of statements (iii) and (iv) is dual to the proof of (i) and (ii), respectively. □

The proofs of Lemma 2.4, Propositions 2.5 and 2.6 are similar to Lemma 2.1, Propositions 2.2 and 2.3, respectively.

**Lemma 2.4.** *Let  $G$  be a linearly ordered group. Then for arbitrary elements  $(a, b), (c, d) \in \mathcal{B}^+(G)$  the following conditions are equivalent:*

- (i)  $(a, b) \preccurlyeq (c, d)$  in  $\mathcal{B}^+(G)$ ;
- (ii)  $a^{-1} \cdot b = c^{-1} \cdot d$  and  $a \geq c$  in  $G$ ;
- (iii)  $b^{-1} \cdot a = d^{-1} \cdot c$  and  $b \geq d$  in  $G$ .

**Proposition 2.5.** *Let  $a, b, c, d$  be arbitrary elements of a linearly ordered group  $G$ . Then the following conditions hold:*

- (i)  $(a, b) = (a, c)(x, y)$  for  $(x, y) \in \mathcal{B}^+(G)$  if and only if  $(c, b) \preccurlyeq (x, y)$  in  $\mathcal{B}^+(G)$ ;



- (ii)  $(a, b) = (x, y)(d, b)$  for  $(x, y) \in \mathcal{B}^+(G)$  if and only if  $(a, d) \preceq (x, y)$  in  $\mathcal{B}^+(G)$ ;
- (iii)  $(a, b) = (a, c)(x, y)(d, b)$  for  $(x, y) \in \mathcal{B}^+(G)$  if and only if  $(c, d) \preceq (x, y)$  in  $\mathcal{B}^+(G)$ .

**Proposition 2.6.** *Let  $a, b, c, d$  be elements of a linearly ordered group  $G$ . Then the following statements hold:*

- (i) *if  $a < c$  in  $G$  then the equation  $(a, b) = (c, d)(x, y)$  has no solutions in  $\mathcal{B}^+(G)$ ;*
- (ii) *if  $a > c$  in  $G$  then the equation  $(a, b) = (c, d)(x, y)$  has the unique solution  $(x, y) = (a \cdot c^{-1} \cdot d, b)$  in  $\mathcal{B}^+(G)$ ;*
- (iii) *if  $b < d$  in  $G$  then the equation  $(a, b) = (x, y)(c, d)$  has no solutions in  $\mathcal{B}^+(G)$ ;*
- (iv) *if  $b > d$  in  $G$  then the equation  $(a, b) = (x, y)(c, d)$  has the unique solution  $(x, y) = (a, b \cdot d^{-1} \cdot c)$  in  $\mathcal{B}^+(G)$ .*

Later we need the following proposition which follows from formula (2) and describe right and left principal ideals in the semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$ .

**Proposition 2.7.** *Let  $a$  be arbitrary element of a linearly ordered group  $G$ . Then the following conditions hold:*

- (i)  $(a, a)\mathcal{B}(G) = \{(x, y) \in \mathcal{B}(G) : x \geq a \text{ in } G\}$ ;
- (ii)  $(a, a)\mathcal{B}^+(G) = \{(x, y) \in \mathcal{B}^+(G) : x \geq a \text{ in } G\}$ ;
- (iii)  $\mathcal{B}(G)(a, a) = \{(x, y) \in \mathcal{B}(G) : y \geq a \text{ in } G\}$ ;
- (iv)  $\mathcal{B}^+(G)(a, a) = \{(x, y) \in \mathcal{B}^+(G) : y \geq a \text{ in } G\}$ .

### 3. ON TOPOLOGIZATIONS OF SEMIGROUPS $\mathcal{B}(G)$ AND $\mathcal{B}^+(G)$

It is obvious that every left (right) topological group  $G$  with an isolated point is discrete. This implies that every countable Baire left (right) topological group is a discrete space, too. Later we shall show that the similar statement holds for Baire semitopological semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$  in the case when  $G$  is a countable linearly ordered group.

For an arbitrary element  $(a, b)$  of the semigroup  $(\mathcal{B}(G), \tau)$  (resp.,  $(\mathcal{B}^+(G), \tau)$ ) we denote

$$\uparrow_{\preceq}(a, b) = \{(x, y) \in \mathcal{B}(G) : (a, b) \preceq (x, y)\} \quad (\text{resp., } \uparrow_{\preceq}(a, b) = \{(x, y) \in \mathcal{B}^+(G) : (a, b) \preceq (x, y)\}).$$

**Lemma 3.1.** *Let  $G$  be a linearly ordered group and  $\tau$  be a topology on  $\mathcal{B}(G)$  ( $\mathcal{B}^+(G)$ ) such that  $(\mathcal{B}(G), \tau)$  ( $(\mathcal{B}^+(G), \tau)$ ) contains an isolated point. Then the space  $(\mathcal{B}(G), \tau)$  ( $(\mathcal{B}^+(G), \tau)$ ) is discrete.*

*Proof.* We consider the case of the semigroup  $\mathcal{B}(G)$ . The proof in the case of  $\mathcal{B}^+(G)$  is similar.

Suppose that  $(a, b)$  is an isolated point of the topological space  $(\mathcal{B}(G), \tau)$ . Now, for an arbitrary  $u \in G$  there exists  $c \in G$  such that  $u > c$ . Since  $G$  is a linearly ordered group,  $d = c \cdot u^{-1} \cdot b < b$  in  $G$ . By Proposition 2.3(iv) the equation  $(a, b) = (x, y)(c, d)$  has the unique solution

$$(x, y) = (a, b \cdot d^{-1} \cdot c) = (a, b \cdot (c \cdot u^{-1} \cdot b)^{-1} \cdot c) = (a, b \cdot b^{-1} \cdot u \cdot c^{-1} \cdot c) = (a, u)$$

in  $\mathcal{B}(G)$ . Then the continuity of right translations in  $\mathcal{B}(G)$  implies that  $(a, u)$  is an isolated point of the topological space  $(\mathcal{B}(G), \tau)$  for arbitrary  $u \in G$ .

Fix an arbitrary element  $v$  of the group  $G$ . Then there exists  $d \in G$  such that  $d < v$ . Since  $G$  is a linearly ordered group,  $c = d \cdot v^{-1} \cdot a < a$  in  $G$ . Then by Proposition 2.3(ii) the equation  $(a, u) = (c, d)(x, y)$  has the unique solution

$$(x, y) = (a \cdot c^{-1} \cdot d, u) = (a \cdot (d \cdot v^{-1} \cdot a)^{-1} \cdot d, u) = (a \cdot a^{-1} \cdot v \cdot d^{-1} \cdot d, u) = (v, u)$$

in  $\mathcal{B}(G)$ . Since  $(a, u)$  is an isolated point of  $(\mathcal{B}(G), \tau)$  the continuity of left translations in  $\mathcal{B}(G)$  implies that  $(v, u)$  is an isolated point of the topological space  $(\mathcal{B}(G), \tau)$  for arbitrary  $u \in G$ . This completes the proof of the lemma.  $\square$

**Theorem 3.2.** *Let  $G$  be a countable linearly ordered group and  $\tau$  be a  $T_1$ -Baire topology on  $\mathcal{B}(G)$  ( $\mathcal{B}^+(G)$ ) such that  $(\mathcal{B}(G), \tau)$  ( $(\mathcal{B}^+(G), \tau)$ ) is a semitopological semigroup. Then the topological space  $(\mathcal{B}(G), \tau)$  ( $(\mathcal{B}^+(G), \tau)$ ) is discrete.*

*Proof.* By Proposition 1.30 of [29] every countable Baire  $T_1$ -space contains a dense subspace of isolated points, and hence the space  $(\mathcal{B}(G), \tau)$   $((\mathcal{B}^+(G), \tau))$  contains an isolated point. Then we apply Lemma 3.1.  $\square$

Theorem 3.2 implies the following corollary:

**Corollary 3.3.** *Let  $G$  be a countable linearly ordered group and  $\tau$  be a Čech complete (locally compact)  $T_1$ -topology on  $\mathcal{B}(G)$   $(\mathcal{B}^+(G))$  such that  $(\mathcal{B}(G), \tau)$   $((\mathcal{B}^+(G), \tau))$  is a semitopological semigroup. Then the topological space  $(\mathcal{B}(G), \tau)$   $((\mathcal{B}^+(G), \tau))$  is discrete.*

**Remark 3.4.** Let  $\mathbb{R}$  be the set of reals with usual topology. It is obvious that  $S_{\mathbb{R}} = \mathbb{R} \times \mathbb{R}$  with the semigroup operation

$$(a, b) \cdot (c, d) = \begin{cases} (a - b + c, d), & \text{if } b < c; \\ (a, d), & \text{if } b = c; \\ (a, b - c + d), & \text{if } b > c, \end{cases}$$

is isomorphic to the semigroup  $\mathcal{B}((\mathbb{R}, +))$ , where  $(\mathbb{R}, +)$  is the additive group of reals with usual linear order. Then simple verifications show that  $S$  with the product topology  $\tau_p$  is a topological inverse semigroup (also, see [32, 33]). Then the subspace  $S_{\mathbb{Q}} = \{(x, y) \in S_{\mathbb{R}} : x \text{ and } y \text{ are rational}\}$  with the induced semigroup operation from  $S$  is a countable, non-discrete, non-Baire topological inverse subsemigroup of  $(S, \tau_p)$ . Also, the same we get in the case of subsemigroup  $S_{\mathbb{Q}}^+ = \{(x, y) \in S_{\mathbb{Q}} : x \geq 0 \text{ and } y \geq 0\}$  of  $(S, \tau_p)$  (see [2, 3, 4, 5, 6]). The above arguments show that the statement of Theorem 3.2 is not true in the case of countable topological inverse semigroups  $\mathcal{B}(G)$  and  $\mathcal{B}^+(G)$ .

Recall [24] a linearly ordered group  $G$  is said to be *densely ordered* if for every positive element  $g \in G$  there exists a positive element  $h \in G$  such that  $h < g$ .

**Remark 3.5.** It is obviously that for a linearly ordered group  $G$  the following conditions are equivalent:

- (i)  $G$  is not densely ordered;
- (ii) for every  $g \in G$  there exists a unique  $g^+ \in G$  such that  $G^+(g) \setminus G^+(g^+) = \{g\}$ ;
- (iii) for every  $g \in G$  there exists a unique  $g^- \in G$  such that  $G^-(g) \setminus G^-(g^-) = \{g\}$ , where  $G^-(g)$  is the *negative cone* on the element  $g$ , i.e.,  $G^-(g) = \{x \in G : x \leq g\}$ .

Later for an arbitrary element  $g$  of a linearly ordered group  $G$  which is not densely ordered by  $g^+$  (resp.,  $g^-$ ) we denote the minimum (resp., maximum) element of the set  $G^+(g) \setminus \{g\}$  (resp.,  $G^-(g) \setminus \{g\}$ ).

**Theorem 3.6.** *Let  $G$  be a linearly ordered group which is not densely ordered. Then every Hausdorff topology  $\tau$  on the semigroup  $\mathcal{B}(G)$   $(\mathcal{B}^+(G))$  such that  $(\mathcal{B}(G), \tau)$   $((\mathcal{B}^+(G), \tau))$  is a semitopological semigroup is discrete, and hence  $\mathcal{B}(G)$   $(\mathcal{B}^+(G))$  is a discrete subspace of any semitopological semigroup which contains  $\mathcal{B}(G)$   $(\mathcal{B}^+(G))$  as a subsemigroup.*

*Proof.* We consider the case of the semigroup  $\mathcal{B}(G)$ . The proof in the case of  $\mathcal{B}^+(G)$  is similar.

We fix an arbitrary idempotent  $(a, a)$  of the semigroup  $\mathcal{B}(G)$  and suppose that  $(a, a)$  is a non-isolated point of the topological space  $(\mathcal{B}(G), \tau)$ . Since the maps  $\lambda_{(a,a)} : \mathcal{B}(G) \rightarrow \mathcal{B}(G)$  and  $\rho_{(a,a)} : \mathcal{B}(G) \rightarrow \mathcal{B}(G)$  defined by the formulae  $((x, y)) \lambda_{(a,a)} = (a, a)(x, y)$  and  $((x, y)) \rho_{(a,a)} = (x, y)(a, a)$  are continuous retractions we conclude that  $(a, a)\mathcal{B}(G)$  and  $\mathcal{B}(G)(a, a)$  are closed subsets in the topological space  $(\mathcal{B}(G), \tau)$  (see [18, Exercise 1.5.C]). For an arbitrary element  $b$  of the linearly ordered group  $G$  we put

$$\text{DL}_{(b,b)} [(b, b)] = \{(x, y) \in \mathcal{B}(G) : (x, y)(b, b) = (b, b)\}.$$

Lemma 2.1 and Proposition 2.2 imply that

$$\text{DL}_{(b,b)} [(b, b)] = \uparrow_{\leq} (b, b) = \{(x, x) \in \mathcal{B}(G) : x \leq b \text{ in } G\},$$

and since right translations are continuous maps in  $(\mathcal{B}(G), \tau)$  we get that  $\text{DL}_{(b,b)} [(b, b)]$  is a closed subset of the topological space  $(\mathcal{B}(G), \tau)$  for every  $b \in G$ . Then there exists an open neighbourhood  $W_{(a,a)}$  of the point  $(a, a)$  in the topological space  $(\mathcal{B}(G), \tau)$  such that

$$W_{(a,a)} \subseteq \mathcal{B}(G) \setminus ((a^+, a^+)\mathcal{B}(G) \cup \mathcal{B}(G)(a^+, a^+) \cup \text{DL}(a^-, a^-)).$$

Since  $(\mathcal{B}(G), \tau)$  is a semitopological semigroup we conclude that there exists an open neighbourhood  $V_{(a,a)}$  of the idempotent  $(a, a)$  in the topological space  $(\mathcal{B}(G), \tau)$  such that the following conditions hold:

$$V_{(a,a)} \subseteq W_{(a,a)}, \quad (a, a) \cdot V_{(a,a)} \subseteq W_{(a,a)} \quad \text{and} \quad V_{(a,a)} \cdot (a, a) \subseteq W_{(a,a)}.$$

Hence at least one of the following conditions holds:

- (a) the neighbourhood  $V_{(a,a)}$  contains infinitely many points  $(x, y) \in \mathcal{B}(G)$  such that  $x < y \leq a$  in the group  $G$ ; or
- (b) the neighbourhood  $V_{(a,a)}$  contains infinitely many points  $(x, y) \in \mathcal{B}(G)$  such that  $y < x \leq a$  in the group  $G$ .

In case (a) by Proposition 2.5 we have that

$$(a, a)(x, y) = (a, a \cdot x^{-1} \cdot y) \notin W_{(a,a)},$$

because  $x^{-1} \cdot y \geq e$  in  $G$ , and in case (b) by Proposition 2.5 we have that

$$(x, y)(a, a) = (a \cdot y^{-1} \cdot x, a) \notin W_{(a,a)},$$

because  $y^{-1} \cdot x \geq e$  in  $G$ , which contradicts the separate continuity of the semigroup operation in  $(\mathcal{B}(G), \tau)$ . The obtained contradiction implies that the set  $V_{(a,a)}$  is singleton, and hence the idempotent  $(a, a)$  is an isolated point of the topological space  $(\mathcal{B}(G), \tau)$ .

Now, we apply Lemma 3.1 and get that the topological space  $(\mathcal{B}(G), \tau)$  is discrete.  $\square$

Theorem 3.6 implies the following three corollaries:

**Corollary 3.7.** *Let  $G$  be a linearly ordered group which is not densely ordered. Then every Hausdorff topology  $\tau$  on the semigroup  $\mathcal{B}(G)$  ( $\mathcal{B}^+(G)$ ) such that  $(\mathcal{B}(G), \tau)$  ( $(\mathcal{B}^+(G), \tau)$ ) is a topological semigroup is discrete, and hence  $\mathcal{B}(G)$  ( $\mathcal{B}^+(G)$ ) is a discrete subspace of any Hausdorff topological semigroup which contains  $\mathcal{B}(G)$  ( $\mathcal{B}^+(G)$ ) as a subsemigroup.*

**Corollary 3.8** ([19]). *Every Hausdorff topology  $\tau$  on the bicyclic extended semigroup  $\mathcal{C}_{\mathbb{Z}}$  such that  $(\mathcal{C}_{\mathbb{Z}}, \tau)$  is a semitopological semigroup is discrete, and hence  $\mathcal{C}_{\mathbb{Z}}$  is a discrete subspace of any Hausdorff semitopological semigroup which contains  $\mathcal{C}_{\mathbb{Z}}$  as a subsemigroup.*

**Corollary 3.9** ([11, 17]). *Every Hausdorff topology  $\tau$  on the bicyclic monoid  $\mathcal{C}(p, q)$  such that  $(\mathcal{C}(p, q), \tau)$  is a semitopological semigroup is discrete, and hence  $\mathcal{C}(p, q)$  is a discrete subspace of any Hausdorff semitopological semigroup which contains  $\mathcal{C}(p, q)$  as a subsemigroup.*

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